THE UNIMAGINED:
CATALOGUES AND THE BOOK OF SAND IN THE *LIBRARY OF BABEL*

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INTRODUCTION.

We adore chaos because we love to produce order.
M. C. Escher

It's an ironic joke that Borges would have appreciated: I am a mathematician who, lacking Spanish, perforce reads *The Library of Babel* in translation. Furthermore, although I bring several thousand years of theory to bear on the story, none of it is literary theory!

Having issued these caveats, it is my purpose to apply mathematical analysis to two ideas in the short story. My goal in this task is not to reduce the story in any capacity; rather it is to enrich and edify by glossing the intellectual margins and substructures. I submit that because of his well-known affection for mathematics, exploring the story through the eyes of a mathematician is a dynamic, useful, and necessary addition to the body of Borgesian thought.

In the first section, *Information Theory: Cataloging the Collection*, I consider the possible forms a Catalogue for the Library might take. Then, in *Real Analysis: The Book of Sand*, I apply elegant ideas from the 17th century and counter-intuitive ideas of the 20th century to the "Book of Sand" described in the final footnote of the story. Three variations of the Book, springing from three different interpretations of the phrase "infinitely thin," are outlined.

I request an indulgence from the reader. This introduction is written in the friendly and confiding first-person singular voice. I will inhabit the first-person plural for the duration of the mathematical expositions. This should not be construed as a "royal we." It has been a construct of the community of mathematicians for centuries and it traditionally signifies two ideas: That "we" are all in consultation with each other through space and time, making use of each other's insights and ideas to advance the on-going human project of mathematics; and, that "we"—the author and reader—are together following the sequences of logical ideas that lead to inexorable, and sometimes poetic, conclusions.

INFORMATION THEORY: CATALOGING THE COLLECTION.

It is a very sad thing that nowadays there is so little useless information.
Oscar Wilde: *A Few Maxims for the Instruction of the Over-Educated*

Information theory is one of the youngest, fastest growing fields in mathematics, essentially born in 1948 when Claude Shannon published *A Mathematical Theory of Communication*. As such, as a discipline, it is still inchoate, still in need of a crystallization into a weltanschauung. For the purposes of this article, we'll say that information theory is the study of compressing and communicating complex information. We consider each book in the Library to be a complex piece of information and our inquiry takes the form of investigating how a catalogue of the Library might encode information about the content and location of books. It strikes us that since the story was written while Borges was engaged in cataloguing the collection of the Miguel Cané Municipal Library, questions of this nature may have taken on additional significance for him.

Typically, a library catalogue card, either physical or virtual, contains two distinct kinds of information. The first sort uniquely specifies a book in such a way that a reader with partial or incomplete information still might identify the book: A title, author, edition, publisher, city of publication, year of publication, and short description of the contents generally appear on a card and prove sufficient. An ISBN also uniquely specifies a book, but probably isn't much use in finding a book if we remembered only a part of it.

The second type of information uniquely specifies a location in the library, although additional knowledge is usually required. For example, under most systems of cataloguing currently in use, the call numbers, in addition to uniquely specifying a book, include an abundance of letters and digits, often interspersed with decimal points. If one does not know, say, where the PQ books are shelved, the information is degraded. Even if the books were arranged alphabetically by author or title, for a large enough collection, we'd still need to know in what general region to begin our search. By analogy, many dictionaries have thumb-nail indentations which enable readers quickly to find a several-letter section. Both of these categories of information are problematic for the Library of Babel.

A form a catalogue might take in principle is: Book (identifiers), Hexagon (location), Shelf (only twenty per hexagon), Position on Shelf (only thirty-two books per shelf). Perhaps surprisingly, self-referentiality is not a problem. A volume of the catalogue, say the tenth, residing in Hexagon thirty-nine, Shelf twenty, Position fourteen, could well be marked on the spine: Catalogue Volume Ten, and correctly describe itself as the tenth volume of the catalogue and specify its location in Position fourteen, on Shelf twenty, in Hexagon thirty-nine. There is no paradox. However, beginning with the obvious, here are some of the difficulties that arise.

Clearly, the Library holds far too many books to be listed in one volume; any catalogue would necessarily consist of a vast number of volumes, which, perversely, are apt to be scattered throughout the Library. Indeed, reminiscent of the approach of *The Approach to Al-Mu'tasim* and the lines in *The Library of Babel*, "[…] in order to locate book A, first consult book B which will indicate the location of A; in order to locate book B, first consult book C, and so on ad infinitum

[…]" (*Ficciones*, 85), an immortal librarian trying to track down a specific book likely has a better chance by making an orderly search of the entire library rather than finding a true catalogue entry for the book. Every plausible entry from any plausible candidate catalogue volume would have to be tracked down, including regressive scavenger hunts. Our immortal librarian would spend a lot of time traversing the Library, oscillating back and forth between different books.

After revealing the nature of the Library, Borges has the librarian note that contained in the Library is "…the faithful catalogue of the Library, thousands and thousands of false catalogues, a demonstration of the fallacy of these catalogues, a demonstration of the fallacy of the true catalogue…" (F, 83). This, then, is the second problem of any catalogue: the only way to verify its faithfulness would be to look up each book. Furthermore, the likelihood of any book being located within a distance walkable within the life-span of a mortal librarian is, to all intents and purposes, *zero*. Sadly, even if we were fortunate enough to possess a true catalogue entry for our "Vindication," presumably our Vindication would merely give details of the death we encountered while spending our life walking in a fruitless attempt to obtain the Vindication.

Let's consider the first category of information found on library cards, that which uniquely specifies the book. *Authorship is moot.* One might argue that the God(s), or the Builder(s) of the Library, is (are) the author(s) of any book. One could make the Borgesian argument that One Man is the author of *all* books. One might also make a claim that the author is a form of a short software program which would, given time and resources, generate all possible variations of twenty-five orthographic symbols in strings of length 1,312,00. Certainly there are many, many[1] books whose first page reads

[1] How many? Precisely $25^{1,308,800}$, which is approximately $10^{1,829,623}$ books. This is an enormous number, yet the odds of randomly selecting one of these books is 1 in $25^{3,200}$, approximately 1 in $10^{4,474}$, which means, essentially, that it will never happen. For comparison's sake, the chance of a single ticket winning a lottery is better than 1 in 10^{8}, so finding such a book is equivalent to winning the lottery more than 449 times in a row.

FICCIONES
By
Jorge Luis Borges

Edited and with Introduction
by Anthony Kerrigan

GROVE PRESS
NEW YORK

Figure 1. The first page of many, many books in the Library.

Pierre Menard may as well be attributed authorship of all the books in the Library.

The title is similarly moot, even if the Library did not contain all possible books for each title. Edition, publisher, city of publication, year of publication: All are meaningless. The one possible sort of information of this variety that may prove useful is that of the short description of the contents.[2] Any book published in the last 500 years likely has a short, reasonably reflective description. A book whose contents consist of the letters MCV repeated over and over evidently has a short description. A book whose entire contents are similar to this 80-symbol first line

vnqenneoij.ernreiuhr.nwperiutuytgn orvjvfgioe,nowejfn vroihwrenoi.wergvn wpriv n

almost certainly doesn't have a short description. Or does it? A fascinating area of study in the field of Information Theory concerns the unknowability of whether or not a line such as the one above has some sort of algorithmic description that is shorter than the line itself. Borges seems to have an intimation of this when he writes (F, 86), "I can not combine certain letters, as *dhcmrlchtdj*, which the divine Library has not already foreseen in combination, and which in

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[2] We'll take "short" to mean "half-page or less." It's much more difficult to say what we mean by "description." We'll take it to mean "significantly narrows the possible contents of the book." For example, "The book is utter gibberish, completely random nonsense," doesn't significantly narrow the possible contents of the book. We are aware that this definition is problematic.

one of its secret languages does not encompass some terrible meaning." Perhaps *wrvns drteee* is more concise description of the line! Or perhaps a succinct translation into English is *Call me Ishmael*.

It does no good to excerpt a passage as a short description; titanic numbers of books in the Library will contain the same passage. In an important sense, then, for all languages spoken by or known to human beings, for the vast majority of books in the Library, *the only possible description of the book is the book itself*. This, in turn, leads to a lovely, inescapable, unimagined conclusion:

The Library is its own Catalogue.

Let's restrict the investigation to a slightly more agreeable collection of books: All books whose entire contents cohere, are recognizably in English, and whose first page contains precisely a short title and a one half-page description, both of which accurately reflect the contents.[3] For example, if the first page of a volume of the Library began:

FICCIONES

> The seventeen pieces in *Ficciones* demonstrate the gargantuan powers of imagination, intelligence, and style of one of the greatest writers of this or any other century. Borges sends us on a journey…

(which is from the back-cover description from the 1999 Grove Press edition of *Ficciones*) and was followed by the precise contents of *Ficciones*, and was filled out by the appropriate number of blank pages, then that Library volume would be included in the collection. We are also willing to include books longer than 410 pages, so long as the title page includes reference to an appropriate volume number. This allows, among other things, for the inclusion of this Catalogue

[3] Any rule of selection will have problems. Some associated with this one are: What does it mean to "cohere"? Would a collection of essays on different topics constitute a coherent work? Would William S. Burroughs' infamous novel *Naked Lunch* register as "recognizably English"? What if the book contains a non-English word, such as "Ficciones"? What if the title, as in the case of *Ulysses*, is more allusive than descriptive? Can any description "accurately reflect" the contents of a book? Regretfully, we'll ignore these and other legitimate, interesting concerns.

of Books in English into the putative catalogue we are trying to define, which we may as well call *Books in English*.

This amenable collection of books is designed to enable *Books in English* to include a title and short accurate description of the contents. This nearly accomplishes the first half of the task of a catalogue; although the books aren't uniquely specified, the scope of possibility is greatly constricted. The other half of a catalogue, that of specifying a location, is fraught with different difficulties.

First, and most strongly emphasized by Borges, is the apparent lack of organization in the distribution of books. It is possible that there is an over-arching pattern, but even if so, it would be impossible to deduce it from local information. The librarian's "elegant hope" that the Library is infinite and periodic would provide a godlike observer with a kind of an order for each book; every particular book would have an infinite number of exact copies—unimaginably distant from each other—and these infinite copies would constitute a set of regularly-spaced three-dimensional lattice points.

Finite or infinite, the problem of identifying individual hexagons of the Library is insurmountable. If Library is a finite space, as described in (Bloch), then the number of hexagons is finite. However, since each hexagon holds 640 books, which is approximately $25^{2.007}$ books, more than $25^{1,311,997}$ ($\sim 10^{1,834,095}$) hexagons are required to hold all the Library's books. This means that if one were to attempt to write out a number for each hexagon in our familiar base-10 notation, it would take 1,834,095 *digits*. Now each book in the Library has exactly 1,312,000 slots to fill, and, moreover, the orthographic symbols contain no (recognizable) digits. Writing a number out in words usually uses many more precious slots, for example,

[one million, eight-hundred thirty-four thousand, and ninety-five] versus [1,834,095].

The first bracketed expression takes 63 spaces, while the second needs only nine. For almost every hexagon in the Library, a volume of a hypothetical *Books in English* catalogue could not actually contain the corresponding hexagon number where a book is shelved!

Trying to get around this problem, one may observe that many numbers have shorter expressions, such as $2^{4,781}$, and legitimately wonder if *every* integer might have a remarkably condensed form.

An insuperable problem is that there are many such condensed expressions, including the one above, that need a computer to calculate! More disturbing, though, is an example of a condensed verbal description of a "small" number—only 2,000 digits—that today's best computers are unable to find:

The largest prime number expressible in two thousand digits.[4]

Thus, even if the catalogue entry for *Ficciones* listed the location as

- Hexagon: largest prime number expressible in two thousand digits
- Shelf: fourteen
- Position: seventeen,

the information is as useless to *us* as it is to a librarian.

One might also try using a higher base, such as base-25, to number the hexagons. There are two problems associated with this: first, it would still take all but two slots of a book to list a hexagon number, which suffices to invalidate the usefulness. Second, since each book contains only twenty-five orthographic symbols, each such symbol would have to stand for a digit. So, if one were to write out the hexagon number in base-25 digits, it would usually look like complete gibberish.[5] Such a book would not be "recognizably English"; thus it would not itself be listed in *Books in English*.

What if, like Ireneo Funes, we resolved to work in base-24,000? (F, 113) It would do no good: Funes created his own unique orthographic symbols, that is, names such as *Brimstone*, *Clubs*, and *The Whale* for each number up to 24,000. We are stuck with 25 orthographic symbols. Instead of combining ten digits in various ways in

[4] To know such a number, a brute-force approach would need the complete list of prime numbers that are expressible in 2,000 digits. Including 0, there are exactly$10^{2,000}$ numbers expressible in 2,000 digits, and, by the famous Prime Number Theorem, there are more than $10^{1,996}$ prime numbers smaller than $10^{2,000}$. A slightly more subtle approach using the classical algorithm of Eratosthenes' sieve would still necessitate considerably more than 10^{996} computations, which is currently unimaginably beyond humanity's power. (Part of the beauty of the Prime Number Theorem is that it provides an excellent estimate of how many primes that there are that are smaller than $10^{2,000}$ without explicitly naming a single one!).

[5] This leads to an unpleasant, yet valid, interpretation of the Library: It is the complete listing of all base-25 numbers comprised of exactly 1,312,000 digits.

five places to number between 1 and 24,000, we would need to combine the 25 symbols in a minimum of 4 places to distinguish 24,000 separate numbers. Not only does this not save much space, it unfortunately leads back to the previous dilemma: Writing out the names of the numbers will result in waterfalls of gibberish.

Finally, a potential catalogue entry might take a different tack. It might give coordinates, such as, "Go up ninety-seven floors, move diagonally left four thousand hexagons and then move diagonally right another two hundred and twenty." The same sorts of problems arise, for most hexagons are unimaginably far away. The example provided above works simply because the numbers involved—97, 4000, and 220—are so miniscule, so accessible. The Library is neither.

The Library is its own Catalogue. Any other Catalogue is unimaginable.

REAL ANALYSIS: THE BOOK OF SAND

To see a World in a Grain of Sand
And a Heaven in a Wild Flower
Hold Infinity in the palm of your hand
And Eternity in an hour.
William Blake: *Auguries of Innocence*

Real Analysis is the branch of mathematics that explores, among other ideas, the nuances of the arbitrarily small. Unimagined and unimaginable consequences await in this section, outgrowths of close examinations of a poetically simple phrase and idea. This includes the application of another recent branch of mathematics, only forty years old, which is not well-known to the non-specialist.

The last footnote to *The Library of Babel* occurs at the very end of the last sentence. It reads:

> Letizia Álvarez de Toledo has observed that the vast Library is useless. Strictly speaking, *one single volume* should suffice: a single volume of ordinary format, printed in nine or ten point type body and consisting of an infinite number of infinitely thin pages. (At the beginning of the 17th century, Cavalieri said that any solid body is the superposition of an infinite number of planes.) This silky vade me-

> cum would scarcely be handy: each apparent leaf would divide into other analogous leaves; the inconceivable central leaf would have no reverse. (F, 88)

Others (Salpeter) have independently noticed that Borges continued to play with the idea of such a book in the short story *The Book of Sand* (*Collected Fictions* 480). The mathematical analysis of a Book of Sand hinges on what is meant by the phrase "infinitely thin pages." Three different interpretations of "infinitely thin" lead to three Books similar in spirit, but different in the details. We'll examine them in ascending order of exoticness.

FIRST INTERPRETATION

If we take "infinitely thin" to mean merely "thinner than any subatomic particle", there are several refreshing possibilities. First, there are $(410)\cdot\left(25^{1,312,000}\right)$ pages in the Library; a very large number, but still finite. Thus, if every page is the same thickness, say

$$\frac{1}{(410)\cdot\left(25^{1,312,000}\right)}\text{th of a meter},$$

then the Book, sans cover, will be exactly one meter thick. This Book, though, would defraud the anonymous librarian of his "elegant hope" that the Library is repeated in its disorder, as well as contravening the explicit statement in the footnote that the book should consist of infinitely many pages. If the pages were to be all the same thickness such as above, then, because of the infinite repetition the book would be infinite in length.

To accomplish infinite repetitions for the Book, we must therefore allow the pages to diminish in size and understand the barest rudiments of the science of infinite sums.[6] Indeed, we will begin by treading parallel to the infinite footfalls, which echo loudly through the ages, of the Paradox of Zeno so beloved by Borges.

Suppose, starting at one end of a room, we were to walk halfway across towards the opposite wall. After a brief pause, we walk half

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[6] Benardete, quoted in (Merrell, 58), independently thinks along similar lines.

the distance from the midpoint towards the opposite wall. After another brief pause, we walk half the distance…

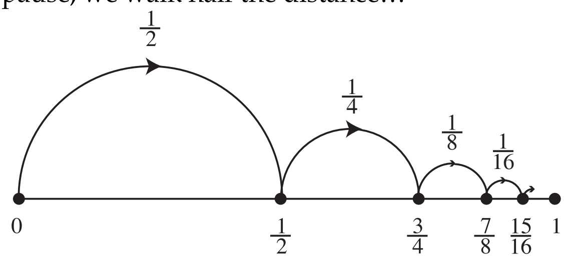


Figure 2. Zeno's walk across the room from 0 to 1

In the coarse world we inhabit, we'll bang into the wall in short order. In the world of mathematics, we can always take another step consisting of half the distance from where we are towards the opposite wall. (Zeno's and Parmenides' paradoxes exploit this chasm between the world of our perceptions and the mathematical idealization of a line segment.)

For the purposes of this article, note that Figure 2 is equivalent to this equation:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots = 1$$

his equation encapsulates a striking fact: by adding up infinitely many partitions, each smaller than the previous by one-half, a form of unity is achieved.

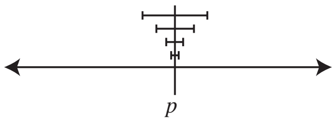
Choose the first page to be one-half of a standard page's thickness, then the next page, half that thickness, the next, half of that thickness, and so on and so on. Then, the entire Book, infinitely repetitive, sans cover, will be exactly *one standard page thickness*. Moreover, after only forty or so pages, every page will be thinner than a proton! Since each successive page is one-half the thickness of the preceding page, all succeeding pages are also thinner than a proton. Moreover, if the initial page is, say, the half of the thickness of a proton, the entire book will be only as thick as a proton. Of course, in this interpretation, although most every page is invisible to the naked eye, it is not the case that any page is actually infinitely thin.

SECOND INTERPRETATION

Here, we take "infinitely thin" in the sense indicated by the reference to Cavalieri's Principle in the footnote: the thickness of a Euclidean plane. The thickness of a plane is the same as the length of a point, which is tricky to define. Consider a point in the line. It is clear that a Euclidean point is thinner than a line segment of *any* positive length. It is somewhat disquieting, though, to say that a point has length 0; if so, how do putting together many 0-length entities create a line of positive length? How could an object be of length 0? A subtle way of evading these traps was crafted at the beginning of the 20th century, primarily through the work of Henri Lebesgue. The theory is now a vast edifice of ideas with ramifications permeating much of modern mathematics; fortunately, we need only a small cornerstone of the theory, considered in the one-dimensional case: the idea of a set of measure 0.

Recall that the real number line consists of all rational and irrational numbers, each representing a point on the line, the distance from the origin to the point. Let **S** be any set contained in the real number line. One says that **S** is *a set of measure 0* if **S** can be contained in a union, possibly infinite, of closed intervals whose lengths add up to an arbitrarily small number.[7] Several examples will help clarify this definition.

Example 1. A single point p in the real number line. Clearly p can be contained in a closed interval of arbitrarily small length. Thus p is a set of measure 0. Note the fine distinction: we are not saying "the point p is of length 0"; we are saying that p is a set whose measure is 0. It turns out—and we'll see an example soon—that there are sets of measure 0 which are quite counter-intuitive.



[7] A possible confusion at this juncture is that we are explicitly identifying the "length of an interval" with a "number." Again, a real-world idea, that of length, interpenetrates a mathematical idealization. We inhabit this limbo for the rest of this section.

Figure 3. An arbitrarily small interval may contain the point *p*.

Example 2. Three points *a*, *b*, and *c* in the real number line. Let

a be contained in an interval of length 1/2,
b be contained in an interval of length 1/4, and
c be contained in an interval of length 1/8

(It doesn't matter if the intervals overlap.) Then the three points are contained in a union of intervals whose sum-length is

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=\frac{7}{8}.$$

Not arbitrarily small yet! But now, let

a be contained in an interval of length 1/4,
b be contained in an interval of length 1/8, and
c be contained in an interval of length 1/16.

Then, since each interval is half the length of its corresponding predecessor, the sum is also halved!

$$\frac{1}{4}+\frac{1}{8}+\frac{1}{16}=\frac{7}{16}.$$

If we play this game again, starting with an interval of length 1/8, we find that

$$\frac{1}{8}+\frac{1}{16}+\frac{1}{32}=\frac{7}{32}.$$

If we continue to put *a*, *b*, and *c* in intervals half of the length of the previous go-round, the triple of intervals will also sum to half the preceding length: First 7/64, then 7/128, and so on. By starting with sufficiently small intervals, we ensure the sum of the three intervals is arbitrarily small—that is, the set **S** = {*a*, *b*, *c*} is a set of measure 0.

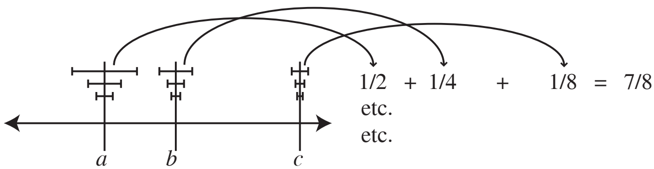


Figure 4. Here, we sum ever-smaller triples of intervals. The sum of each triple is half the length of the preceding triple's sum.

Example 3. It is a curious fact that it is difficult to show that the interval of numbers between 1 and 4 is *not* of measure 0. Certainly our intuition informs us that the minimum length of intervals necessary to cover [1,4] will sum to 3, but demonstrating it rigorously is a highly non-trivial exercise.

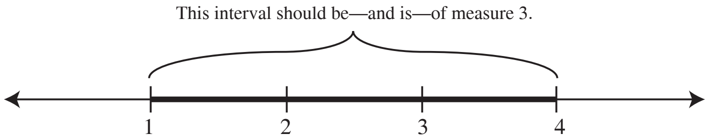


Figure 5. The measure of the interval from 1 to 4.

Back to the infinitely thin pages of the Book. We interpret "infinitely thin" as meaning that each page has a thickness of measure 0. We also assume, as we did at the end of the first interpretation, that within this Tome, the books of the Library repeat over and over, enacting the anonymous librarian's "elegant hope" of a repeating order. We are therefore confronted with an intriguing problem:

There are infinitely many pages, each of which has thickness of measure 0. How thick is the Book?

Such a question deserves a exhilarating answer:

The thickness of the Book is of measure 0!

In other words, if we looked at the Book sideways, we would not be able to see it, let alone be able to open it. How does this unimaginable state of affairs arise? Surprisingly, once we know to look for it, it is relatively easy to understand and believe.

The goal is to show that the thickness of the Book can be contained in an arbitrarily small interval, for if this can be done, then by the definition, the Book is of measure 0. We'll accomplish this by covering the thickness of each page in ever-smaller intervals in a sneaky way that exploits Zeno's paradox. Let

- the thickness of the first page be contained in an interval of length 1/2,
- the thickness of the second page be contained in an interval of length 1/4,
- the thickness of the third page be contained in an interval of length 1/8,

- and so on,
- and so on.

We saw in the first interpretation that

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots=1,$$

so the thickness of the Book is contained in an infinite union of intervals which sum to 1. Here's where the sneaky part comes in. Now, let

- the thickness of the first page be contained in an interval of length 1/4,
- the thickness of the second page be contained in an interval of length 1/8,
- the thickness of the third page be contained in an interval of length 1/16,
- and so on,
- and so on.

This time, the infinite union of intervals sums to

$$\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\cdots=\frac{1}{2}.$$

This is seen by simply subtracting 1/2 from both sides of the previous equation.[8] If we start by letting the thickness of the first page be contained in an interval of length 1/8, then the sum becomes:

$$\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}+\cdots=\frac{1}{4}.$$

Clearly, by continuing to play this game of lopping the intervals in half, we can ensure that the infinite union of intervals containing the thickness of the Book will sum to an arbitrarily small number. This means that the thickness of the Book is of measure 0, an outcome surely unimagined by Borges.

How it is possible to create a line segment, a set of positive measure, out of points of measure 0? That is a long story for another day.

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[8] Notice how we are exploiting an aspect of the idea of infinity: We are throwing away a term from the left side of the equation, but still have infinitely many terms to account for the infinite number of pages!

THIRD INTERPRETATION

Perhaps the elusive nature of the preceding interpretation is unsatisfying; we never took a stand on how thick "infinitely thin" is; we merely observed that it is of measure 0. For the third interpretation, we will glimpse some of the rudiments of one of the most underutilized mathematical theories of the 20th century: Non-standard analysis. The roots of the development of non-standard analysis began with Leibniz, one of the inventors of calculus. Both Leibniz and Newton used infinitely small quantities, *infinitesimals* (also known as *fluxions*), in their early calculations. In his foreword to the revised edition of the seminal work of non-standard analysis (Robinson), Luxemburg notes that "Bishop Berkeley disdainfully referred to infinitesimals as the 'ghosts of departed quantities,'" and, that in response to this and other attacks, "Leibniz proposed a program to conceive of a system of numbers that would include infinitesimally small as well as infinitely large numbers."

Because of the difficulties inherent in instantiating the program and for other historical reasons, Leibniz' ideas lay fallow for almost 300 years. In 1961, with the publication of *Non-standard Analysis*, Abraham Robinson rebutted Berkeley and fulfilled Leibniz' dream. Using various tools of logic and set theory developed in the late 19th and early 20th centuries, Robinson was able to create a consistent, logical model of a number system that included infinitesimals.

It should be mentioned, with all due respect, that adherents of non-standard analysis possess an almost religious fervor about the topic. This may be because the mainstream of mathematics has, at least for now, marginalized non-standard analysis due to its non-intuitive constructions and technical complexities. Bearing this in mind, here are selections, excerpted from (McKinzie), of an introduction to a college textbook that approaches calculus from the non-standard viewpoint (emphases added by present author).

> In grade school and high school mathematics, the real number system is constructed gradually in several stages. Beginning with the positive integers, the systems of integers, rational numbers and finally real numbers are built up…

> What is needed [for an understanding of the calculus] is a sharp distinction between numbers which are small enough to be neglected and numbers which aren't. Actually, no real number except zero is small enough to be neglected. *To get around this difficulty, we take the bold step of introducing a new kind of number, which is infinitely small and yet not equal to zero…*
> The real line is a subset of the hyperreal line; that is, each real number belongs to the set of hyperreal numbers. Surrounding each real number *r*, we introduce a collection of hyperreal numbers infinitely close to *r*. *The hyperreal numbers infinitely close to zero are called infinitesimals*. The reciprocals of nonzero infinitesimals are infinite hyperreal numbers. The collection of all hyperreal numbers satisfies the same algebraic laws as the real numbers…
> We have no way of knowing what a line in physical space is really like. It might be like the hyperreal line, the real line, or neither. However, in applications of the calculus it is helpful to imagine a line in physical space as a hyperreal line. The hyperreal line is, like the real line, a useful mathematical model for a line in physical space.
> (Keisler)

In non-standard analysis, there are infinitely many hyperreal infinitesimals clustered around 0, each one smaller than any positive real number. Each signifies an infinitely small distance. We may simply assign any infinitesimal we wish to each page of the Book.[9] By the rules of non-standard analysis, we compute the thickness of the Book by adding together all of the infinitesimals. For a summation such as this one, adding the infinite number of infinitesimals produces yet another infinitesimal, so the Book is, again, infinitely thin: Never to be seen, never to be found, never to be opened. This time, though, we may elegantly console ourselves that the infinite thinness is a precisely calculable non-standard thickness.

Regardless of which interpretation we assume, if the pages are 'infinitely thin,' then by necessity the Book of Sand itself is infinitely thin.

[9] If a mathematically-sophisticated reader is worried about invoking the Axiom of Choice, the issue is easily side-stepped by assigning the same infinitesimal, ß, for the thickness of each page.

CONCLUSION

> *Either a universe that is all order, or else a farrago thrown together at random yet somehow forming a universe. But can there be some measure of order subsisting in yourself, and at the same time disorder in the greater whole?* Marcus Aurelius*: Meditations*

Borges was a master of understating ideas, allowing them the possibility of gathering heft and power, generating their own gravity. I'm under no delusion that he traced out the consequences of the dormant mathematics I uncovered, that there can be no catalogue and that a Book of Sand couldn't be seen, let alone opened or read. I allow myself the ambition, though, to paraphrase what Borges wrote in a forward (Barrenechea, vii) and hope that this article would have taught him something about himself.

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